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CONSTRUCTING FAMILIES OF CONNECTIONS ON $\mathbb{P}^1$ AND τ-DIVISORS

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Abstract

The moduli of connections on trivial vector bundles over $\mathbb{P}^1$

$$\frac{d}{dz}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left(\frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_2}{z-t}\right)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

does not coincide with the space of initial conditions of Okamoto $S \setminus Y_{red}$. There exists a divisor on $S \setminus Y_{red}$ which does not correspond, called a τ-divisor. If we consider the moduli of connections on vector bundles of degree zero over $\mathbb{P}^1$, then it is isomorphic to $S \setminus Y_{red}$. Thus on the τ-divisor the type of the vector bundles jumps. In this poster we explain this phenomenon.

Notation

M : a connected complex analytic manifold
$\mathcal{E}$: a holomorphic bundle on $\mathbb{P}^1 \times M$, $\operatorname{rank}\mathcal{E} = r$, $\deg \mathcal{E}_{|\mathbb{P}^1\times\{m\}} = 0$ $(\forall m \in M)$

$$\begin{array}{ccc} & \mathcal{E} & \\ & \downarrow & \\ & \mathbb{P}^1 \times M & \\ \swarrow & & \searrow \pi \\ \mathbb{P}^1 & & M \end{array}$$

$\nabla : E \to E \otimes \Omega^1_{\mathbb{P}^1}(D)$: a connection on $\mathbb{P}^1$ with pole divisor D

Families of vector bundles on $\mathbb{P}^1$

Theorem 1 (τ-divisor) The support Θ of the sheaf $R^1\pi_*\mathcal{E}(-1)$ is the set of points $m \in M$ such that the restriction of $\mathcal{E}$ to $\mathbb{P}^1 \times \{m\}$ is not trivial. If $\Theta \neq \emptyset$ and $\Theta \neq X$, then Θ is a hypersurface of M.

Theorem 2 (Rigidity of a trivial bundle) If there exists $m^\circ \in M$ such that $\mathcal{E}^\circ := \mathcal{E}_{|\mathbb{P}^1\times\{m^\circ\}}$ is trivial, then there exists an open neighbourhood V of m° such that the restriction of $\mathcal{E}$ to $\mathbb{P}^1 \times V$ is trivial.

Irreducible connections on $\mathbb{P}^1$

Definition 1 (Birkhoff-Grothendieck) For any vector bundle E on $\mathbb{P}^1$ there is an isomorphism $E \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r), a_1 \geq \cdots \geq a_r$. We call $a_1 \geq \cdots \geq a_r$ a type and $\delta(E) := \Sigma_{i=1}^r a_1 - a_i$ a defect of the vector bundle E.

Definition 2 (Irreducibility) A connection (E, ∇) is irreducible if it satisfies for any subbundle F, $\nabla F \subset F \otimes \Omega^1_{\mathbb{P}^1}(D)$

Proposition 1 (Boundness of a defect) For any irreducible connection $\nabla : E \to E \otimes \Omega^1_{\mathbb{P}^1}(D)$, the following inequality holds

$$\delta(E) \leq (\deg D - 2)\frac{r(r-1)}{2}.$$

Later we consider the case $D = 4$, $r = 2$ so the possibble types are

$$\mathcal{E}_{|\mathbb{P}^1\times\{m\}} \simeq \begin{cases} \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \\ \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1). \end{cases}$$

Example Painlevè V $(0, 0, 1)$

$$\nabla = d + \left(\frac{A_0}{z} + \frac{A_1}{z-1} + A_\infty\right)dz = d + A_z\frac{dz}{z(z-1)}$$

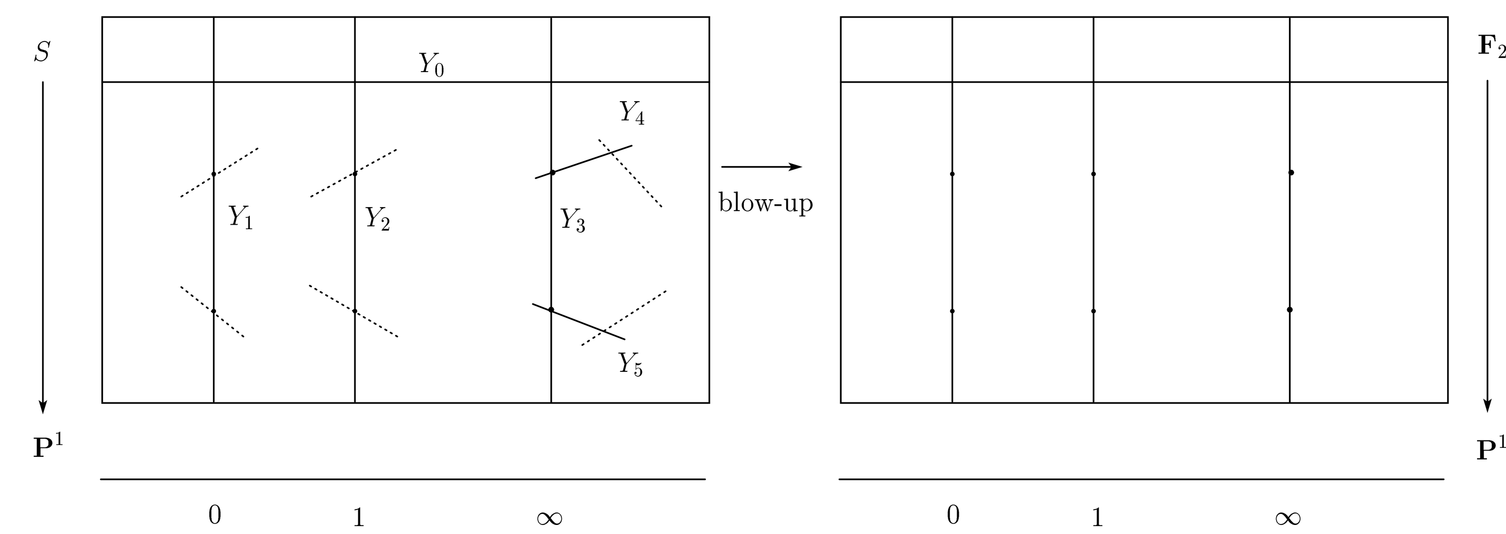
$$A_\infty = \begin{pmatrix} -\frac{t}{2} & 0 \\ 0 & \frac{t}{2} \end{pmatrix}$$

The singularities z	0	1	∞
Katz invariant	0	0	1
generalized local exponents	$\pm\frac{\theta_0}{2}$	$\pm\frac{\theta_1}{2}$	$\pm(\frac{t}{2}z + \frac{\theta_\infty}{2})$

$$A_z = \begin{pmatrix} L & M \\ z-q & -L \end{pmatrix}$$

$$L = -\tfrac{t}{2}z^2 + \tfrac{1}{2}(t - \theta_\infty)z + \tfrac{1}{2}\left(tq^2 - tq + \theta_\infty q + 2p\right)$$

$$M = \frac{(q+z-1)(2p+(q-1)qt)^2+(q-1)(z-1)\theta_0^2-qz\theta_1^2+(q-1)q\theta_\infty(4p+2(q-1)qt+(q-z)\theta_\infty)}{4(q-1)q}$$



$-K_S = Y = 2Y_0 + Y_1 + Y_2 + 2Y_3 + Y_4 + Y_5$
(S, Y) : Okamoto-Painlevé pair

Families of connections on the τ-divisor

We constructed a families of connections on the τ-divisor

$$A_z = \begin{pmatrix} L & M \\ -1 & -L \end{pmatrix}, \quad L = \tfrac{tz^2}{2} + \tfrac{1}{2}(-t + \theta_\infty - 2)z - \tfrac{u_2}{2},$$

$$M = \tfrac{1}{4}\left({u_2}^2 - \theta_0^2\right) + \tfrac{1}{4}z\left(\theta_0^2 - \theta_1^2 - (2u_2 - \theta_\infty + 2)(\theta_\infty - 2)\right),$$

where the variable u_2 is the coordinate on the τ-divisor.

Parabolic structures

Fixing the generalized local exponents determinds the moduli space of connections. But to constract a family of connections we fixed the data $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ and parabolic structure l_∞. This causes a jumping phenomenon of the underlying vector bundles and the τ-divisor depends on it.

parabolic structure (cyclic vector) $\Longleftrightarrow$ τ-divisor

References

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[PSa] van der Put, M. and M.-H. Saito, Moduli spaces for linear differential equations and the Painlevé equations